Ques If e_1 and e_2 be the eccentricity of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Then show that $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

Soln for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

we know that $b^2 = a^2(e_1^2 - 1)$

$\Rightarrow b^2 = a^2 e_1^2 - a^2$

$\Rightarrow a^2 e_1^2 = a^2 + b^2$

$\Rightarrow e_1^2 = \frac{a^2 + b^2}{a^2}$ ——— (1)

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$\Rightarrow a^2 = b^2(e_2^2 - 1) \Rightarrow a^2 = b^2 e_2^2 - b^2$

$\Rightarrow e_2^2 = \frac{a^2 + b^2}{b^2}$ ——— (2)

From (1) & (2) we get

$\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}$

$= 1$

Proved

Ques Chords of circle $x^2+y^2=a^2$ touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

P.T. their middle point lie on the curve $(x^2+y^2)^2 = a^2x^2 - b^2y^2$

Soln Let the eqn of the chord $x^2+y^2=a^2$ have middle point (α, β)

then the eqn becomes

$$x\alpha + y\beta = \alpha^2+\beta^2 \quad \text{--- (1)}$$

$\because$ this chord touches the hyperbola at a single point it is also a tangent to the hyperbola. Let the tangent touch the hyperbola at point (x_1, y_1)

$\therefore$ the eqn of the tangent is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{--- (2)}$$

If we put $x_1 = a\sec\theta$ and $y_1 = b\tan\theta$ then (2) implies

$$\frac{x \cdot a\sec\theta}{a^2} - \frac{y\, b\tan\theta}{b^2} = 1$$

$$\Rightarrow \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \quad \text{--- (3)}$$

$\because$ Eqn (1) & (3) are same

$$\Rightarrow \frac{\alpha a}{\sec\theta} = \frac{\beta b}{-\tan\theta} = \frac{\alpha^2+\beta^2}{1}$$

$$\Rightarrow \sec\theta = \frac{\alpha a}{\alpha^2+\beta^2} \quad \& \quad \tan\theta = \frac{-\beta b}{\alpha^2+\beta^2}$$

$$\therefore \sec^2\theta - \tan^2\theta = \frac{a^2\alpha^2}{(\alpha^2+\beta^2)^2} - \frac{\beta^2 b^2}{(\alpha^2+\beta^2)^2}$$

$$\Rightarrow 1 = \frac{a^2\alpha^2 - \beta^2 b^2}{(\alpha^2+\beta^2)^2}$$

$$\Rightarrow (\alpha^2+\beta^2)^2 = a^2\alpha^2 - b^2\beta^2$$

$\therefore$ the locus of the middle point is $(x^2+y^2)^2 = a^2x^2 - b^2y^2$.

Proved

Ques P.T. the locus of the middle points of normal chords of the rectangular hyperbola $x^2-y^2=a^2$ is $(y^2-x^2)^3 = 4a^2x^2y^2$.

Soln The equation of the normal at a point (x_1, y_1) is given by

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{-y_1/b^2}$$

$\because$ In this case the equation of hyperbola is $x^2-y^2=a^2$ we take point as $(a\sec\theta, a\tan\theta)$

$$\Rightarrow \frac{x - a\sec\theta}{\sec\theta/\cancel{a}} = \frac{y - a\tan\theta}{-\tan\theta/\cancel{a}}$$

$$\Rightarrow \frac{x}{\sec\theta} - a = \frac{-y}{\tan\theta} + a$$

$$\Rightarrow \frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a \quad \text{—— (1)}$$

Again eqn to the chord of given hyperbola whose middle point is (α, β) is given by

$$x\alpha - y\beta = \alpha^2 - \beta^2 \quad \text{——— (2)}$$

$\because$ Eqn (1) & (2) represent the same line we get by comparing.

$$\alpha \sec\theta = -\beta \tan\theta = \frac{\alpha^2 - \beta^2}{2a}$$

$$\Rightarrow \sec\theta = \frac{\alpha^2 - \beta^2}{2a\alpha} \quad \& \quad \tan\theta = -\frac{(\alpha^2 - \beta^2)}{-2a\beta}$$

$$\Rightarrow \sec^2\theta - \tan^2\theta = \frac{(\alpha^2 - \beta^2)^2}{4a^2\alpha^2} - \frac{(\alpha^2 - \beta^2)^2}{4a^2\beta^2}$$

$$\Rightarrow 1 = \frac{(\alpha^2 - \beta^2)^2}{4a^2}\left\{\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right\}$$

$$\Rightarrow 1 = -\frac{(\alpha^2 - \beta^2)^2}{4a^2}\left(\frac{\alpha^2 - \beta^2}{\alpha^2\beta^2}\right)$$

$$\Rightarrow 4a^2\alpha^2\beta^2 = -(\alpha^2 - \beta^2)^3$$

$\therefore$ The locus of (α, β) is

$$-4a^2x^2y^2 = (\cancel{-}x^2 + y^2)^3$$

Proved